

# Extremal spacings of random unitary matrices

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## Abstract

Extremal spacings between unimodular eigenvalues of random unitary matrices of size  $N$  pertaining to circular ensembles are investigated. Probability distributions for the minimal spacing for various ensembles are derived for  $N = 4$ . We argue that for large matrices the average minimal spacing  $s_{\min}$  of a random unitary matrix behaves as  $N^{-1/(1+\beta)}$  for  $\beta$  equal to 0, 1 and 2 for circular Poisson, orthogonal and unitary ensembles, respectively. For these ensembles also asymptotic probability distributions of minimal spacing  $s_{\min}$  are numerically studied and the statistics of the largest spacing  $s_{\max}$  are investigated.

## 1 Introduction

Random unitary matrices are useful to describe spectra of periodic quantum systems the classical analogues of which are chaotic [1, 2]. The choice of a specific ensemble of matrices is dictated by symmetry properties of the investigated physical system. If the system possess no time-reversal symmetry the *circular unitary ensemble* (*CUE*) of matrices distributed according tho the Haar measure of the unitary group is appropriate [3]. For systems with a generalized time reversal symmetry the *circular orthogonal ensemble* (*COE*) describes properly statistical properties of spectra if we neglect additional subtleties caused by specific rotational symmetry features of systems with half-integer spin which are of no concern for investigations reported in this paper. In the case of classically regular dynamics the spectrum of the evolution operator displays level clustering characteristic to the *circular Poissonian ensemble* (*CPE*) of diagonal random unitary matrices. To describe intermediate statistics one uses interpolating ensembles of unitary matrices [4, 5, 6] or composed ensembles of unitary matrices [7]. In the case of emerging chaos, in which the chaotic layer covers only a fraction of the phase space of the

classical system one may apply the distribution of Berry and Robnik, originally used for autonomous systems [8].

To characterize statistical properties of spectra of random matrices one often uses the distribution of nearest neighbour spacings  $s$ , also called level spacing distribution, with density as usual denoted by  $P$  [3, 9]. The random variable  $s$  is the distance between adjacent eigenphases normalized to make the mean spacing equal to unity.

In this work we investigate the distribution of yet another random variable – the minimal spacing between two eigenphases  $s_{\min}$ . As the above mentioned distribution of spacings also  $s_{\min}$  encodes interesting information about the spectrum. Indeed, observe, e.g., that for any unitary matrix  $U$  the size of its minimal spacing  $s_{\min}$ , provides an information, to which extent the matrix  $U$  is close to be degenerated. For completeness we are also going to study the size of the largest spacing  $s_{\max}$  defined similarly.

The distribution of extremal eigenvalues of random Hermitian matrices was analyzed several years ago [10]. Our current approach is in some sense analogous, as we investigate the extremal gaps between eigenvalues of a unitary matrix distributed along the unit circle. After a part of our project was completed we learned about a relevant paper of Arous and Bourgade [11], in which the distribution of extremal spacings was studied for random matrices of circular unitary ensemble.

The paper is organized as follows. For exemplary cases of random matrices of size  $N = 4$  we derive in Section 2 exact forms of minimal spacing distributions. The chosen dimension allows exact calculations, which become prohibitively complicated for larger matrices and, on other hand, is the minimal dimension in which results for *CUE* and *CPE* can be compared with those for the ensemble consisting of tensor products of two *CUE* random matrices of size  $N = 2$ . Such an ensemble corresponds to a generic local dynamics in a two-qubit system [12].

The case of large matrices is studied in Section 3. In particular, we argue that for a random unitary matrix of size  $N$  the size of the minimal gap scales as  $s_{\min} \approx N^{-\frac{1}{1+\beta}}$  where  $\beta = 0, 1$  and  $2$  for the Poissonian, orthogonal and unitary circular ensemble, respectively. Using known formulae for the level spacing distributions we show how to estimate the mean maximal spacings for the three ensembles. Finally we formulate plausible conjectures concerning the asymptotic (in  $N$ ) distributions of the minimal spacing  $s_{\min}$  for all three ensembles and compare them with numerical results.

We use the following notation. For a single unitary or orthogonal matrix  $A$  of size  $N$  we shall consider its spectrum  $\{\exp(i\varphi_j)\}_{j=1}^N$ , where  $(\varphi_1, \dots, \varphi_N)$  represents the vector of the eigenphases ordered nondecreasingly,  $0 \leq \varphi_1 \leq \dots \leq \varphi_N < 2\pi$ . We order nondecreasingly the spacings  $\varphi_2 - \varphi_1, \dots, \varphi_N - \varphi_{N-1}, 2\pi + \varphi_1 - \varphi_N$  between neighboring eigenphases and denote obtained sequence by

$$s_{\min} := s_1 \leq \dots \leq s_N =: s_{\max}. \quad (1.1)$$

The standard level spacing distribution  $P$  is given by the average  $\frac{1}{N} \sum_{m=1}^N P_m$  where  $P_m$  is the density of the variable  $s_m$ .

## 2 Case study: minimal spacings for $N = 4$

Our goal is to derive the exact probability distribution of the minimal spacing  $P_{\min}$  for exemplary ensembles of random unitary matrices of size  $N = 4$ . Apart of the Poissonian and

the unitary ensemble we analyze also the tensor product of two independent random matrices of size  $N = 2$ . This ensemble, denoted by  $CUE_2 \otimes CUE_2$ , describes dynamics of two independent quantum sub-systems [12].

To derive the desired distribution we calculate the tail distribution  $T(t) = \mathbb{P}(s_{\min} > t)$  and take the derivative of  $T$ . We have

$$\begin{aligned} T(t) &= \mathbb{P}(s_{\min} > t) = \mathbb{P}(\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \varphi_4 - \varphi_3, 2\pi + \varphi_1 - \varphi_4 > \pi t/2) \\ &= \int_{\{\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \varphi_4 - \varphi_3, 2\pi + \varphi_1 - \varphi_4 > \pi t/2\}} P^{\text{ord}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4), \end{aligned} \quad (2.1)$$

where  $P^{\text{ord}}$  is the joint probability distribution of ordered eigenphases, which can be obtained from the joint probability distribution for a given ensemble. After changing variables,  $\psi_1 = \varphi_1, \psi_2 = \varphi_2 - \varphi_1, \psi_3 = \varphi_3 - \varphi_2$  and  $\psi_4 = \varphi_4 - \varphi_3$ , the integration domain splits into two tetrahedrons. Simple but long calculations yield in each case the tail distribution function  $T$ , which leads to the probability density  $P_{\min} = -\frac{d}{dt}T$ .

(a)  $CUE_2 \otimes CUE_2$

$$\begin{aligned} P_{\min}(s_{\min}) &= \frac{1}{4\pi} \left( 2\pi(1 - s_{\min})(4 - \cos(\pi s_{\min}/2)) - 3\sin(\pi s_{\min}/2) \right. \\ &\quad \left. + 8\sin(\pi s_{\min}) - 3\sin(3\pi s_{\min}/2) \right) \end{aligned} \quad (2.2)$$

(b)  $CUE_4$

$$\begin{aligned} P_{\min}(s_{\min}) &= \frac{1}{72\pi^2} \sin^2(\pi s_{\min}/4) \left( 666 + 720\pi^2(1 - s_{\min})^2 \right. \\ &\quad + 36(11 + 16\pi^2(1 - s_{\min})^2) \cos(\pi s_{\min}/2) \\ &\quad + 18(8\pi^2(1 - s_{\min})^2 - 13) \cos(\pi s_{\min}) - 100 \cos(3\pi s_{\min}/2) \\ &\quad - 608 \cos(2\pi s_{\min}) - 380 \cos(5\pi s_{\min}/2) + 234 \cos(3\pi s_{\min}) \\ &\quad + 74 \cos(7\pi s_{\min}/2) - 58 \cos(4\pi s_{\min}) + 10 \cos(9\pi s_{\min}/2) \\ &\quad + 24\pi(1 - s_{\min}) \left( 60 \sin(\pi s_{\min}/2) + 63 \sin(\pi s_{\min}) + 22 \sin(3\pi s_{\min}/2) \right. \\ &\quad \left. \left. + 2 \sin(2\pi s_{\min}) - 4 \sin(5\pi s_{\min}/2) \right) \right) \end{aligned} \quad (2.3)$$

(c)  $CPE_4$

$$P_{\min}(s_{\min}) = 3(1 - s_{\min})^2. \quad (2.4)$$

The graphs of these functions are presented in Figure 2.1. The behavior around zero of the densities encodes information about level repulsion and level clustering. The variable  $s_{\min}$  is the smallest distance between two neighboring eigenphases. Therefore, the fact that its density is separated from zero, say  $P_{\min} > 1$  around zero, means that for small  $\epsilon > 0$  the probability that some two phases are in the distance closer than  $\epsilon$  equals  $\mathbb{P}(s_{\min} < \epsilon) = \int_0^\epsilon P_{\min}(s)ds > \epsilon$ . In the cases of  $CPE_4$  and  $CUE_4$  this consistency with the level clustering and repulsion observed in the distribution of spacings  $P$  is clearly visible. The figure shows

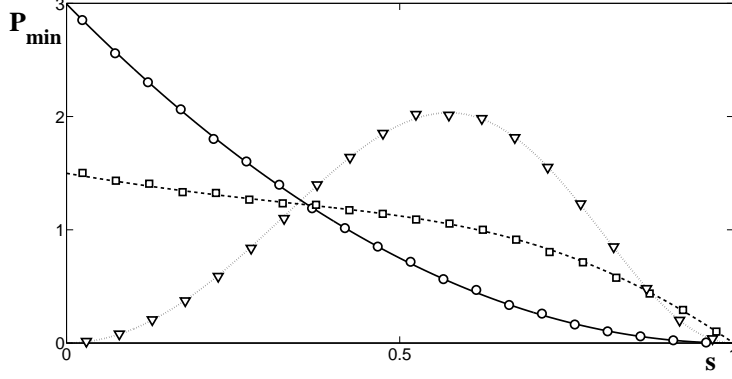


Figure 2.1: Probability densities  $P_{\min}$  for the random matrices of size  $N = 4$  pertaining to (○)  $CPE_4$ , (□)  $CUE_2 \otimes CUE_2$ , (▽)  $CUE_4$ . The symbols denote numerical results obtained for  $2^{14}$  independent matrices, while the curves represent distributions (2.4), (2.3) and (2.2) of the variable  $s_{\min}$ , respectively.

that for  $CUE_2 \otimes CUE_2$  ensemble the eigenphases of the tensor product tend to accumulate in a spectacular contrast to the case a single random unitary matrix form  $CUE$  [12].

Numerical results show that for large  $N$  and  $m$  of order of  $N/2$  the distributions of the variables  $s_m$  are close to the distribution of the level spacing  $s$ . However, for any  $N$  the distributions of the smallest spacing  $s_{\min}$  and of the largest spacing  $s_{\max}$  differ considerably. We shall then analyze these distributions of extremal spacings, which can be used as auxiliary statistical tools to characterize ensembles of random matrices.

### 3 Extremal statistics for large matrices

In this section we analyze extremal gaps in spectra of circular ensembles of random matrices of a large size,  $N \gg 1$ . As usual, we parametrize canonical ensembles by the level repulsion parameter  $\beta$ , equal to 0, 1 and 2 for Poissonian, orthogonal and unitary ensembles respectively. Then, relevant quantities are denoted with the index  $\beta$ , e.g.  $P_\beta$ ,  $\beta = 0, 1, 2$  is the density of the level spacing of the corresponding ensemble.

#### 3.1 Mean minimal spacing

To get an estimation of the behavior of the mean minimal spacing with the matrix size let us assume that spacings  $s_j$ ,  $j = 1, \dots, N$  are independent random variables. For small spacing one has  $P_\beta(s) \sim s^\beta$ , so the integrated distribution  $I_\beta(s) = \int_0^s P_\beta(s') ds'$  behaves as  $I_\beta(s) \sim s^{1+\beta}$ . A matrix  $U$  of size  $N$  yields  $N$  spacings  $s_j$ . Thus the minimal spacing  $s_{\min}$  occurs on average for such an argument of the integrated distribution that  $I_\beta(s_{\min}) \approx 1/N$ .

This implies that  $(s_{\min})^{1+\beta} \approx 1/N$ , which allows us to estimate the average minimal spacing

$$\langle s_{\min} \rangle \approx N^{-\frac{1}{1+\beta}}. \quad (3.1)$$

For CUE, i.e. in the case  $\beta = 2$  this statement is consistent with the rigorous results [11] of Arous and Bourgade. As shown in Fig. 3.1 the above heuristic reasoning provides the correct value of the exponent in dependence of the mean minimal spacing  $\langle s_{\min} \rangle$  on the matrix size  $N$  for *CPE* ( $\beta = 0$ ), *COE* ( $\beta = 1$ ) and *CUE* ( $\beta = 2$ ).

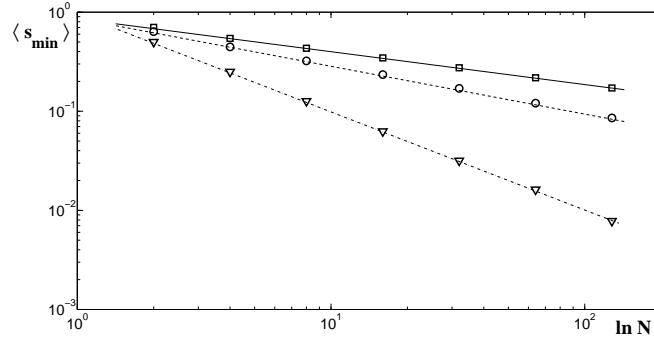


Figure 3.1: Mean minimal spacing  $\langle s_{\min} \rangle$  as a function of the matrix size  $N = 2^m$  for  $(\nabla)$  *CPE*,  $(\square)$  *COE* and  $(\circ)$  *CUE* and  $m = 1, \dots, 7$ . Symbols denote numerical results obtained for  $2^{14}$  independent random matrices. Solid, dashed and dash-dot lines are plotted with slopes implied by the estimation (3.1) and equal to  $-1$ ,  $-1/2$  and  $-1/3$ , respectively. Linear fit to numerical data yields the slopes  $-0.98$ ,  $-0.48$ ,  $-0.33$ , respectively.

### 3.2 Mean maximal spacing

We are going to estimate the average maximal spacing  $\langle s_{\max} \rangle$  for random unitary matrices of the circular orthogonal ensemble. Matrix of size  $N$  yields  $N$  spacing  $s_j$ . In analogy to the previous reasoning we shall assume that all spacings are independent random variables described by the Wigner surmise

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}. \quad (3.2)$$

Thus the integrated distribution  $I(s) = \int_0^s P(s') ds'$  reads  $I(s) = 1 - e^{-\pi s^2/4}$ . The maximal spacing  $s_{\max}$  occurs on average for such an argument of the integrated distribution function that  $1 - I(s_{\max}) \approx 1/N$ . This implies that  $e^{-\pi s_{\max}^2/4} \approx 1/N$ , which allows us to estimate the average maximal spacing,

$$\langle s_{\max} \rangle_{COE}^2 \approx \frac{4}{\pi} \ln N. \quad (3.3)$$

This implies that  $\langle s_{\max} \rangle^2$  grows with the matrix size  $N$  proportionally to  $\frac{4}{\pi} \ln N$  what is demonstrated in Fig. 3.2.

Now let us deal with the case of the circular unitary ensemble. Here we employ the Wigner formula for the level spacing distribution  $P$  of a large  $CUE$  matrix  $P(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}$ . By the same reasoning as above we obtain an estimate that the maximal spacing  $s_{\max}$  occurs on average for such an argument of the integrated distribution function  $I(s) = \int_0^s P(s') ds'$  that  $1 - I(s_{\max}) \approx 1/N$ . Thus

$$\frac{1}{N} \approx \int_{s_{\max}}^{\infty} \frac{32}{\pi^2} s^2 e^{-4s^2/\pi} ds. \quad (3.4)$$

We change the variable  $u = 4s^2/\pi$  and obtain  $\frac{1}{N} \approx \int_{4s_{\max}^2/\pi}^{\infty} \frac{2}{\sqrt{\pi}} u^{1/2} e^{-u} du$ . Therefore, supposing  $s_{\max}$  is large we get

$$\frac{1}{N} \approx \frac{4}{\pi} s_{\max} e^{-4s_{\max}^2/\pi}. \quad (3.5)$$

Now we take the logarithm of both sides, neglect  $\ln s_{\max}$  as it is of lower order than  $s_{\max}^2$  for large  $s_{\max}$ , and arrive at

$$\langle s_{\max} \rangle_{CUE}^2 \approx \frac{\pi}{4} \ln N. \quad (3.6)$$

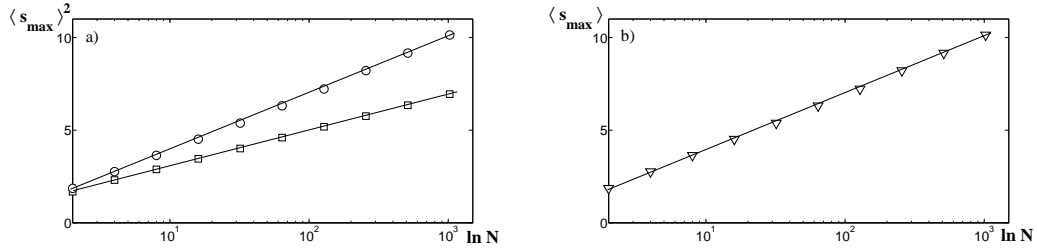


Figure 3.2: Mean maximal spacing  $\langle s_{\max} \rangle$  as a function of the matrix size  $N = 2^m$  for a) ( $\square$ )  $COE$  and ( $\circ$ )  $CUE$ , b) ( $\nabla$ )  $CPE$  and  $m = 0, \dots, 10$ . Symbols denote numerical results obtained for  $2^{14}$  independent random matrices. Solid, dashed and dash-dot lines are plotted with slopes implied by the estimation (3.5), (3.8), (3.9) and equal to  $4/\pi$ , 1 and  $\pi/4$ , respectively. Linear fit to numerical data yields the slopes 1.32, 0.97, 0.84, respectively.

In the case of a Poissonian spectrum the level spacing distribution displays an exponential tail,  $P(s) \sim e^{-s}$ . Thus the integrated distribution function  $I(s) = \int_0^s P(s') ds'$  behaves as  $I(s) = 1 - e^{-s}$ . For a matrix of size  $N$  the maximal spacing  $s_{\max}$  occurs on average for such an argument that  $1 - I(s_{\max}) \approx 1/N$ . This implies that  $e^{-s_{\max}} \approx 1/N$  and enables us to estimate the average maximal spacing for the circular Poisson ensemble as

$$\langle s_{\max} \rangle_{CPE} \approx \ln N. \quad (3.7)$$

Analyzing estimations from equations (3.3), (3.6) and (3.7) gives the slopes  $A_{COE} = \frac{4}{\pi} \approx 1.27$ ,  $A_{CUE} = \frac{\pi}{4} \approx 0.77$  and  $A_{CPE} = 1$ , which are comparable with numerical results  $A_{COE} \approx 1.33$ ,  $A_{CUE} \approx 0.84$  and  $A_{CPE} \approx 0.97$ .

### 3.3 Distribution of extremal spacings

To study the distributions of the minimal spacing  $P_{\min}$  introduce a rescaled variable suggested by (3.1),

$$x_{\min}^{(\beta)} := A_{(\beta)} N^{\frac{1}{1+\beta}} s_{\min}, \quad (3.8)$$

where  $A_{(\beta)}$  is a constant, in general different for *CPE*, *COE* and *CUE*.

The case of the unitary ensemble was recently studied by Arous and Bourgade [11], who derived the following expression for the asymptotic distribution of the minimal spacing

$$P_{\min}(x_{\min}) = 3x_{\min}^2 e^{-x_{\min}^3}, \quad (3.9)$$

in the rescaled variable  $x_{\min} = (\pi/3)^{2/3} N^{1/3} s_{\min}$ . This result suggests the following general form of the distribution of minimal spacing for all three ensembles considered labeled by the level repulsion parameter  $\beta$

$$P_{\min}^{(\beta)}(x_{\min}) := (\beta + 1) x_{\min}^{\beta} e^{-x_{\min}^{\beta+1}}, \quad (3.10)$$

which agree perfectly with the numerical data.

The above formula has a structure  $F(x) := \frac{df(x)}{dx} e^{-f(x)}$ , which helps to determine the normalization. Numerical results show that constants read:  $A_{(0)} = 1$  for *CPE*,  $A_{(1)} = \langle s \rangle$  for *COE*, and  $A_{(2)} = (\pi/3)^{2/3}$  for *CUE*.

Returning to the original variable we obtain the distributions  $P_{\min}^{(\beta)}(s_{\min})$ ,

$$P_{\min}^{(0)} = A_{(0)} N e^{-N s_{\min}}, \quad (3.11)$$

$$P_{\min}^{(1)} = 2A_{(1)}^2 N s_{\min} e^{-A_{(1)}^2 N s_{\min}^2}, \quad (3.12)$$

$$P_{\min}^{(2)} = 3A_{(2)}^3 N s_{\min}^2 e^{-A_{(2)}^3 N s_{\min}^3}. \quad (3.13)$$

The distributions of the minimal spacing obtained numerically for Poisson, orthogonal and unitary circular ensembles of random matrices of the size  $N = 100$  are presented in Fig. 3.3.

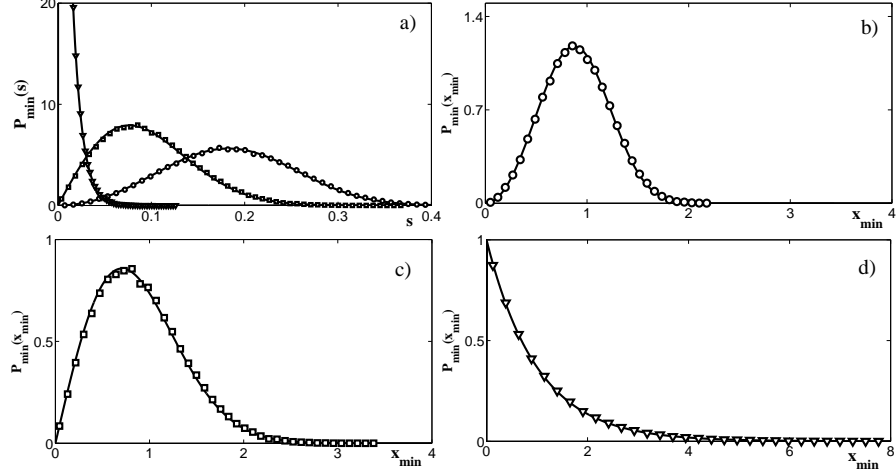


Figure 3.3: Probability distributions  $P_{\min}$  for random unitary matrices of *CUE* ( $\nabla$ ), *COE* ( $\square$ ), and *CPE* ( $\circ$ ) in normal scale (fig. a)), and rescaled for b) *CUE*, c) *COE* and d) *CPE*. Symbols denote numerical results obtained for  $2^{17}$  independent matrices of size  $N = 100$ , while solid curves represent asymptotic predictions.

## 4 Concluding remarks

A significant and spectacular difference between the Poissonian ensemble on one side and *COE* and *CUE* on the other, *vis.* degree of “repulsion” between adjacent levels can be effectively analyzed in terms of distributions of the minimal spacings. We analyzed the average minimal spacing and postulated a general form of its distribution for the classical ensembles. For *CUE* this distribution coincides with the one recently derived by Arous and Bourgade [11]. We support our findings by numerical simulations showing a perfect match with the theoretical predictions.

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